Multivariable Analysis

Zambelli Lorenzo
BSc Applied Mathematics

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1 Introduction

This notes are based on the material of the Lecture's notes and the course textbook.

2 Derivatives

Definition 1 *Let $f : U \to \mathbb{R}^m$ be given where U is an open subset of $\mathbb{R}^n$. The function f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear trasformation and*

$$f(p+v) = f(p) + T(v) + R(v) \Rightarrow \lim_{|v|\to 0} \frac{R(v)}{\|v\|} = 0$$

We say that the Taylor remainder R is subliear because it tends to 0 faster than $\|v\|$.

Remark: Df is the total derivative or Frechet derivative and if the function is differentiable at U then the map $x \mapsto (Df)_x$ defines a function

$$Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$

Theorem 2 *If f is differentiable at p then it unambiguosly determines $(Df)_p$ according to the limit fomrula, valid for all $u \in \mathbb{R}^n$,*

$$(Df)_p(u) = \lim_{t\to 0} \frac{f(p+tu) - f(p)}{t}$$

Definition 3 *If f is differentiable at p, then for all basis vector $e_i \in \mathbb{R}^n$ (orthonormal),*

$$\left.\frac{\partial f_i}{\partial x_j}\right|_p = \lim_{t\to 0} \frac{f_i(p+te_j) - f_i(p)}{t}$$

are the ij^{th} partial derivative of f at p if the limit exists.

Definition 4 (Jacobian Matrix) *If f is differentiable (in coordinates: $f = f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$, then*

$$(Df)_p = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where the rows of $Df|_p$ are the transpose of the gradient of f_i at p for all $i \in \{1, ..., m\}$ ($\nabla^T f_i(p)$)

Corollary 5 *If the total derivative exists then the partial derivatives exist and they are the entries of the matrix that represents the total derivative*

Remark: Do not confuse the total derivative $Df|_p$ with the direction derivatives of f at $p \in U$ which is the limit, if exists

$$\nabla_p f(u) = (Df)_p(u) = \lim_{t\to 0} \frac{f(p+tu)-f(p)}{t}$$

If the i,j-th partial derivatives of f at p exist for all $i \in \{1,...,m\}$, then together they form the directional derivative of f in this specific e_i direction.

Remark: If f is differentiable, then

$$\nabla_p f(u) = \nabla f(p)\cdot u = \frac{\partial f}{\partial x_1}u_1 + \cdots + \frac{\partial f}{\partial x_n}u_n$$

Proposition 6 *Let $\mathbb{R}^n$ and two norm $\|\cdot\|_a$, $\|\cdot\|_b$, then*

$$\exists r_1, r_2 > 0 \quad s.t.\ \forall v \quad r_1\|v\|_a \leq \|v\|_a \leq r_2\|v\|_b$$

Theorem 7 *Differemtiability implies continuity*

Theorem 8 *If the partial derivatives of $f : U \to \mathbb{R}^m$ exist and are continuous then f is differentiable.*

Theorem 9 *Let f and g be differentiable. Then*

(a) $D(f + cg) = Df + cDg$

(b) $D(constant) = 0$ *and* $D(T(x)) = T$ *where T is a linear map.*

(c) $D(g \circ f) = Dg \circ Df$ *Chain Rule*

(d) $D(fg) = Dfg + fDg$ *Leibniz Rule*

Theorem 10 *A function $f : U \to \mathbb{R}^m$ is differentiable at $p \in U$ if and only if each of its components f_i is differentiable at p. Furthermore, the derivative of its i^{th} component is the i^{th} component of the derivative*

Theorem 11 (Mean Value Theorem) *If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on U and the segment $[p, q]$ is contained in U then*

$$\|f(q) - f(p)\| \leq M\|q - p\|$$

where $M = \sup\{\|(Df)_x\| : x \in (p, q) \subset U\}$.

Theorem 12 (C^1 Mean Value theorem) *If $f : U \to \mathbb{R}^m$ is of class C^1 (its derivative exists and is continuous) and if the segment $[p, q] \subset U$ then*

$$f(q) - f(p) = \int_0^1 (Df)_{p+t(q-p)}\, dt(q - p)$$

where the integral is the average derivative of f on the segment. Note that conversely it holds too.

Corollary 13 *Assume that U is connected and open. If $f : U \to \mathbb{R}^m$ is differentiable and for each point $x \in U$ we have $(Df)_x = 0$ then f constant.*

3 Higher Derivatives

The derivative $D^k f$ $\forall k \in \mathbb{N}$ is the same sort of thing that f, namely a function from a open subset of a vector space into another vector space.

Definition 14 *Assume $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in U, then f is second differentiable at $q \in U$ if $Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is differentiable at $q \in U$*

Remark: The second derivative at p is a linear map from $\mathbb{R}^n$ into $\mathcal{L}$. For each $v \in \mathbb{R}^n$, $(D^2 f)_p(v)$ belongs to $\mathcal{L}$ and therefore is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ so $(D^2 f)_p(v)(w)$ is bilinear and we write it as $(D^2 f)_p(v, w)$. The higher derivatives are defined in the same way.

Remark: If f second-differentiable on U then $x \mapsto (D^2 f)_x$ defines a map

$$D^2 f : U \to \mathcal{L}^2 = \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)) \cong \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)$$

where $\mathcal{L}^2$ is the vector space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

Remark: Let

$$\|f(v)\| = \sup \left\{ \frac{\|f\| \|v\|}{\|v\|} : v \in \mathbb{R} \right\}$$

then

$$\|Df(v)\| \leq \|Df\| \|v\|$$
$$\|D^2 f(v)\| \leq \|D^2 f\| \|v\|^2$$
$$\|D^k f(v)\| \leq \|D^k f\| \|v\|^k \ k \in \mathbb{N}$$

Theorem 15 *If $(D^2 f)_p$ exists then $(D^2 f_k)_p$ exists, the second partials at p exist, and*

$$(D^2 f_k)_p(e_i, e_j) = \frac{\partial^2 f_k(p)}{\partial x_i \partial x_j}$$

Conversely, existence of the second partials implies existence of $(D^2 f)_p$, provided that the second partials exist at all points $x \in U$ near p and are continuous at p

Theorem 16 *If $(D^2 f)_p$ exists then it is symmetric: for all $v, w \in \mathbb{R}^n$ we have*

$$(D^2 f)_p(v, w) = (D^2 f)_p(w, v)$$

Corollary 17 *Corresponding mixed second partials of a second-differentiable function are equal,*

$$\frac{\partial^2 f_k(p)}{\partial x_i \partial x_j} = \frac{\partial^2 f_k(p)}{\partial x_j \partial x_i}$$

Corollary 18 *If f is differentiable on U, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist on U and are continuous at p, then*

$$\frac{\partial^2 f^k}{\partial x_i \partial x_j} = \frac{\partial^2 f^k}{\partial x_j \partial x_i} \quad \forall i, j, k$$

Corollary 19 *The r^{th} derivative, if it exists, is symmetric: Permutation of the vectors $v_1, ..., v_r$ does not effect the value of $(D^r f)_p(v_1, ..., v_r)$. Corresponding mixed higher-order partials are equal.*

3.1 Smoothness class

Definition 20 *$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k on U if $f, Df, D^2f, ..., D^kf$ exist on U and D^kf is continuous on U*

Definition 21 *$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is of class C^∞ if $f \in C^k$ $\forall k \in \mathbb{N}$*

Corollary 22 *$f \in C^k$ (or C^∞) iff all partial derivatives up to order k (or for all partial derivatives) exist and are continuous.*

Consider the set $C^k(U, \mathbb{R}^m)$ of C^k maps on U, for which the following norm is bounded

$$\|f\|_{C^k} := \max_{0 \le i \le k} \sup_{x \in U} \|D^i f|_x\|$$

Theorem 23 *$(C^k(U, \mathbb{R}^m), \|\cdot\|_{C^k})$ is a Banach space for all $k < \infty$. A sequence of functions $f_n \in C^k(U, \mathbb{R}^m)$ converges to $f \in C^k(U, \mathbb{R}^m)$ in $\|\cdot\|_{C^k}$ iff*

$$f_n \rightrightarrows f, \cdots, D^k f_n \rightrightarrows D^k f$$

on U (uniform converges of f and its differentials up to order k)

Corollary 24 ($C^k - M$ test) *Let $f_n \in C^k(U, \mathbb{R}^m)$ be such that $\|f_n\|_{C^k} \le a_n$, where $\sum_{n=1}^\infty a_n$ converges. Then $\sum_{n=1}^\infty f_n$ converges to a function $f \in C^k(U, \mathbb{R}^m)$. Moreover, for all $s \le k$:*

$$D^2 f = \sum_{n=1}^\infty D^2 f_n$$

term by term differentiable is valid for all $s \le k$.

4 Taylor's theorem

Theorem 25 (Taylor's theorem) *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be of class C^N on U. Let $[p, p+v] \subset U$. then*

$$f(p+v) = f(p) + \sum_{k=1}^{N-1} \frac{1}{k!} D^k f|_p(\underbrace{v...v}_{k\,times}) + R_{N-1}(f, v)$$

where

$$R_{N-1}(f, v) = \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} D^N f|_{p+tv}(v...v) dt$$

Remark: When $N = 1$, we get the C^1 mean value theorem

Corollary 26 *Under the assumptions of the theorem,*

$$f(p+v) = f(p) + \sum_{k=1}^{N-1} \frac{1}{k!} D^k f|_p(\underbrace{v...v}_{k\,times}) + o(\|v\|^N)$$

where $o(\|v\|^n) = f(v) \Leftrightarrow f(v)/\|v\|^n \to 0$ as $\|v\|^n \to 0$

Remark: Let $x = v + p$ so that $v_i = (x - p)_i$. In two dimension with $x_1 = x$ and $x_2 = y$

$$f(x) = f(p) + \frac{\partial f}{\partial x}(p)(x - x_0) + \frac{\partial f}{\partial y}(p)(y - y_0) + \cdots$$

5 Flat vs Analytic functions

In the previous section we have discuss the Taylor expansion and we learn that given $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (for simplicity $m = 1$) the Taylor's theorem holds up to any order. In general, the series doesn't have to converge. Moreover, if the series does converge, it doesn't have to converge to a given function.

Definition 27 *When a function f have $\forall k \in \mathbb{N}\ D^k f|_0 = 0$ and is smooth, then such f is called flat.*

Definition 28 *A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is (real) analytic if $\forall p_0 = (x_1^0 ... x_n^0) \in U$*

$$f = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} c_{k_1,\cdots,k_n}(x_1 - x_1^0)^{k_1} \cdots (x_n - x_n^0)^{k_n}$$

convergent power series in a neighbourhood of p_0. Alternatively, a function f is (real) analytic on U if $f \in C^\infty$ on U and the Taylor series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1,\cdots,i_k} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} (x_{i_1} - x_{i_1}^0) \cdots (x_{i_k} - x_{i_k}^0)$$

converges to f in a neighbourhood of $p_0 = (x_1^0 ... x_n^0)$ for all $p_0 \in U$ (note that the series are local).

5.1 Relation with complex analysis

Definition 29 *$f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic if f is $\mathbb{R}$ differentiable on $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\frac{\partial f}{\partial \overline{z_j}} = 0$ for all $j = 1, ..., n$ where*

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

$z_j = x_j + iy_j$. This implies that $Df|_p$ is complex linear at every $p \in U$.

Theorem 30 *f is holomorphic on U iff near every point it can be represented by a convergent power series*

Corollary 31 *Holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathbb{R}^n$ are real-analytic ($Ref(Rez_1 ... Rez_n)$ is real analytic). Conversely, a real analytic function $h : \mathbb{R}^n \to \mathbb{R}$ admits (at least locally) a holomorphic extension*

6 Find extrema of a function

Definition 32 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function. It is said to have a local minimum (resp., maximum) at $p_0 \in U$ if $\exists$ a small neighborhood $p_0 \in V \subset U$ such that*

$$f(p) \geq f(p_0), \quad resp. f(p) \leq f(p_0)$$

for all $p \in V$. p_0 is a strict local minimum (resp. maximum) if

$$f(p) > f(p_0), \quad resp. f(p) < f(p_0)$$

for all $p \in V \backslash \{p_0\}$.

Definition 33 *Local minima and maxima are called extrema of a function*

Proposition 34 *Consider a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Assume that $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_2}$ exist at a point $p_0 \in U$. If p_0 is a local extremum of f, then*

$$\left.\frac{\partial f}{\partial x_i}\right|_{p_0} = 0,\, i = 1, ..., n$$

Remark: Points where

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

vanishes are called **critical points**. They don't have to be minima or maxima

Theorem 35 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^2 in a neighbourhood of $p_0 \in U$ which is a critical point of f ($\nabla f|_{p_0} = 0$). If the Hessian*

$$\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{p_0} \in Mat(n \times n, \mathbb{R})$$

is positive (resp., negative) definite, i.e. the eigenvalues are positive (resp. negative) then p_0 is a local minimum (resp., maximum). If the eigenvalues are both positive and negative, then we have a saddle point. Instead, if the eigenvalues are 0 then we do not have enough information to tell.

Remark: To check positive/negative definiteness, one can use Sylvester's criterion from Linear algebra

7 Implicit function theorem

Definition 36 *Two open subsets V_1 and V_2 of $\mathbb{R}^n$ are called C^k(resp.,C^∞)-diffeomorphic if there exists a bijection $f : V_1 \to v_2$ such that f and f^{-1} are of class C^k (resp.,C^∞).*

Remark: If f is a bijection and f and f^{-1} are C^0, then f is called a homeomorphism

Theorem 37 (Implicit Function Theorem) *Let U be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and $F = (f_1, ..., f_m) : U \to \mathbb{R}^m$ be of class $C^k(C^\infty)$, $k \geq 1$, on U. Consider the following equation*

$$F(x, y) = z_0$$

where $z_0 \in \mathbb{R}^m$. If there exists $(x_0, y_0) \in U$ with $F(x_0, y_0) = z_0$ and the $m \times m$ matrix

$$B = \left.\frac{\partial f_i}{\partial y_j}\right|_{(x_0, y_0)}$$

is invertible, then the equation admits a unique solution $y = g(x)$ near (x_0, y_0). Furthermore, g is $C^k(C^\infty)$

Theorem 38 (Implicit Function Theorem 2) *If the mapping $F : U \to \mathbb{R}^n$ defined in a neighborhood U of the point $(x_0, y_0) \in \mathbb{R}^{m+n}$ is such that*

- $F \in C^{(p)}(U, \mathbb{R}^n)$, $p \geq 1$
- $F(x_0, y_0) = 0$
- $F'_y(x_0, y_0)$ *is an invertible matrix*

then there exists am $(m+n)$ dimensional interval $I = I_x^m \times I_y^n \subset U$, where

$$I_x^m = \{x \in \mathbb{R}^m \,\big|\, |x - x_0| < \alpha\} \quad I_y^n = \{y \in \mathbb{R}^n \,\big|\, |y - y_0| < \beta\}$$

and a mapping $f \in C^{(p)}(I_x^m, I_y^n)$ such that

$$F(x, y) = 0 \Leftrightarrow y = f(x)$$

for any point $(x, y) \in I_x^m \times I_y^n$ and

$$f'(x) = -\frac{F'_x(x, f(x))}{F'_y(x, f(x))}$$

Theorem 39 *If $h : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $C^k(C^\infty)$, $k \geq 1$ and $Dh|_{x_0}$ is invertible, then h is a C^k-diffeomorphism near x_0: there exists a small open neighbourhood $x_0 \in U_1 \subset U$ such that $h : U_1 \to U_2 = h(U_1)$ is a C^k-diffeomorphism. In particular, U_2 is open and $h|_{U_1}$ is an open map (for any V an open subset pf U_1, the image $h(V)$ is open)*

8 Banach Fixed point Theorem

Definition 40 (Lipschitz) *A function f is Lipschitz in U w.r.t the variables $x = (x_1, ..., x_n)$ and Lipschitz constant L if*

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for all $x, y \in U$.
Similarly, f is said to be locally Lipschitz in U w.r.t. $x = (x_1, ..., x_n)$ if for every point $x_0 \in U$ there exists a neighbourhood $x_0 \in V \subset U$ such that

$$\|f(x) - f(y)\| \leq L^V\|x - y\|$$

on V. In other words, f is Lipschitz on V

Theorem 41 (Banach Fixed-point Theorem) *Let (M, d) be a complete metric space. Let $f : M \to M$ be such that*

$$d(f(q), f(p)) \leq Kd(q, p)$$

for all $q, p \in M$, where $k < 1$ is a constant not depending on q and $p \in M$. Then f has a unique fixed point $p_0 \in M$, i.e.

$$f(p_0) = p_0 \quad f(p) = p \Rightarrow p = p_0$$

9 Ordinary Differential equations

Definition 42 *Let $t \in \mathbb{R}$ and $F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$ be a function of $n+1$ variables. An ordinary differential equation (ODE) of $n-th$ order is an equation of the form*

$$F(t, x, x', x", ..., x^{(n)}) = 0$$

where t is the independent variable, $x = x(t)$ is a function of t and $x', x", \cdots, x^{(n)}$ are its derivatives.
A function $x = x(t)$ is a solution of the ODE if the substitution of $x(t), x'(t), ..., x^{(n)}(t)$ into F makes the ODE hold identically

Remark: The above equation is implicit and therefore the ODE is said to be in implicit form. An n-th order ODE is siad to be in explicit form if it can be written as follows

$$x^{(n)} = f(t, x, x', ..., x^{(n-1)})$$

Definition 43 *Let $t \in \mathbb{R}$ and $f_i : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$, $i = 1, ...n$, be functions of $n+1$ variables. A first order system of differentiable equations (in explicit form) is a set of n equations*

$$\begin{cases} x_1' = f_1(t, x_1, ..., x_n) \\ \vdots \quad \vdots \\ x_n' = f_n(t, x_1, ..., x_n) \end{cases}$$

Or, in more compact notation,

$$x' = F(t, x), \quad F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$$

A solution of this ODE is a vector function

$$x = x(t) = (x_1(t) \cdots x_n(t))$$

that is differentiable on some interval $t \in (a, b) \subset \mathbb{R}$ and if substitution of $x = x(t)$ into the $x' = F(t, x)$ makes the equality hold trivially.

Definition 44 (Initial value problem) *Initial value problem (IVP) asks for solution of $x' = F(t, x)$ that passes through a given point $(t_0, x_0) \in U \subset \mathbb{R}^{n+1}$, i.e. $x(t_0) = x_0 \in \mathbb{R}^n$. The solution of the IVP is equivalent to the integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} F(t, x(t))dt$$

More precisely, assume $F \in C^0$ and let $x = x(t)$ be a solution, then

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(t)dt = x_0 + \int_{t_0}^{t} F(t, x(t))dt$$

Conversely, if $x = x(t)$ is a continuous solution of

$$x(t) = x_0 + \int_{t_0}^{t} F(t, x(t))dt$$

Then, $x = x(t) \in C^1$, $x(t_0 = x_0$ and $x'(t) = F(t, x(t))$.

9.1 Linear ODEs

Definition 45 *A system of Linear ODEs is an explicit system of ODEs of the following form*

$$x' = A(t)x + B(t)$$

where A is a time dependent $n \times n$ matrix and $B(t) \in \mathbb{R}^n$ is a time dependent vector

Definition 46 *A linear system is called homogeneous if $B(t) = 0$, and otherwise it is called inhomogeneous*

Definition 47 *A linear system is said to have constant coefficients if $A(t) = A(t_0)$ and $B(t) = B(t_0)$, i.e. they do not depend on t*

Theorem 48 *Consider a linear system*

$$x' = A(t)x$$

where $A = A(t) : (a, b) \subset \mathbb{R} \to Mat(n \times n, \mathbb{R})$ is continuous. Then the set of solutions is a vector space isomorphic to $\mathbb{R}^n$.

Theorem 49 *Consider a linear system $x' = A(t)x + B(t)$, where $A = A(t) : (a, c) \subset \mathbb{R} \to Mat(n \times n, \mathbb{R})$ and $B = B(t) : (a, c) \subset \mathbb{R} \to \mathbb{R}^n$ are C^0. Assume $x^{inh} = x^{inh}(t)$ is a solution of the inhomogeneous system and $x^h = x^h(t)$ is an arbitrary solution of the homogeneous sytem $x' = A(t)x$. Then, $x = x^h(t) + x^{inh}(t)$ is a solution of $x' = A(t)x + B(t)$*

Remark: The difference between two solutions of the inhomogeneous system is a solution of the homogeneous one.

Proposition 50 *Consider an IVP*

$$x' = Ax \quad x(0) = x_0 \in \mathbb{R}^n$$

where A has constant coefficients. Then it can be exactly solved and the solution has the form

$$x(t) = e^{At}x_0$$

where e^{At} is the exponential of an $n \times n$ matrix At, defined by series

$$\exp\{At\} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

Remark: To solve a particular ODE, it is helpful to use the Jordan decomposition of A. For more info look in the lecture notes of Linear system

Theorem 51 (Existence and uniqueness) *Let $F = F(t, x) : U \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous on U and locally Lipschitz on U w.r.t $x = (x_1, ..., x_n)$. If $(t_0, x_0) \in U$, then the IVP*

$$\begin{cases} x' = F(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution, which can be extended to the boundary of U

Theorem 52 *Let $F : (d, c) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Take a segment $[a, b] \subset [d, c]$ and assume that F is globally Lipschitz on $[a, b] \times \mathbb{R}^n$ w.r.t $x \in \mathbb{R}^n$. Then the IVP*

$$\begin{cases} x' = F(t, x) \\ x(a) = x_0 \end{cases}$$

has a unique solution defined on $[a, b]$.

Corollary 53 *Consider a linear system of $1 - st$ order ODEs $x' = A(t)x + B(t)$, where $A : (d, c) \to Mat(n \times n, \mathbb{R})$ and $B : (d, c) \to \mathbb{R}^n$ are continuous. Then the IVP, $x(t_0) = x_0$ where $t_0 \in (d, c)$ has a unique solution on (d, c) for all initial conditions $x_0 \in \mathbb{R}^n$*

Theorem 54 (Peano existence theorem) *If $F : U \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on U, then every IVP*

$$\begin{cases} x' = F(t, x) \\ x(t_0) = x_0 \end{cases}$$

where $(t_0, x_0) \in U$ has a (possibly non-unique) solution.

Theorem 55 (Separations of variables for 1-d ODEs) *Consider an ordinary differential equation of the form*

$$x' = g(x)f(t)$$

where $g : U \subset \mathbb{R} \to \mathbb{R}$ is C^0 and non-zero on U and $f : V \subset \mathbb{R} \to \mathbb{R}$ is C^0. Then every initial value problem

$$\begin{cases} x' = g(x)f(t) \\ x(t_0) = x_0 \end{cases}$$

where $(t_0, x_0) \in U \times V$. has a unique local solution, which can moreover be obtained by solving

$$\int_{x_0}^{x} \frac{dx}{g(x)} = \int_{t_0}^{t} f(t)dt$$

for x as a function of t

Definition 56 *Any set of n linearly independent solutions $x_1(t), ..., x_n(t)$ is called a fundamental system of solutions; the matrix $X(t)$, where $X(t) = (x_1(t), ..., x_n(t))$, is also called a fundamental system or a fundamental matrix.*

Remark: Any solution $x(t)$ of $x' = A(t)x$ cen be written as

$$x(t) = X(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

where c_i are constant for all $i \in \{1, ..., n\}$.

Remark: Every fundamental matrix $X = X(t)$ solves the matrix equation

$$X' = A(t)X$$

and that it can be written as

$$X(t) = X^E(t) \cdot C$$

where C is a constant and non-degenerate $n \times n$ matrix and $X^E(t)$ is the unique solution of matrix IVP

$$(X^E)' = A(t)X^E, \quad X^E(t_0) = E$$

where E is the identity matrix.

Definition 57 *Let $Y = Y(t)$ be a solution to the matrix equation $X' = A(t)X$. The (time-dependent) determinant of $Y(t)$ is called the Wronskian determinant or Wronskian of $Y(t)$.*

Theorem 58 *Let $A = A(t)$ be continuous and let $X = X(t)$ be a solution of the matrix equation $X' = A(t)X$. Then $X(t)$ is a fundamental matrix iff the Wronskian $w(t)$ of $X(t)$ is non-zero. Moreover, $w(t)$ satisfies the differential equation*

$$w' = (trA(t))w$$

where $trA(t)$ is the trace of $A(t)$. Hence,

- $w(t) = w(t_0) \exp\left\{\int_{t_0}^t tr(A(s))ds\right\}$
- $det(X^E(t)) = \exp\left\{\int_{t_0}^t tr(A(s))ds\right\}$

where $X^E(t)$ solves $X' = A(t)X$, $X(t_0) = E$.
In particular, $w(t)$ is either identically 0 or it is non-zero for every t

9.2 Variation of constant

Consider an inhomogeneous system

$$x' = A(t)x + B(t)$$

where

$$A : (c, d) \to Mat(n \times n, \mathbb{R})$$
$$B : (c, d) \to \mathbb{R}^n$$

are continuous. Let $X(t)$ be the fundamental matrix for the homogeneous equation $x' = A(t)x$. In the variation of constant method, the constants in

$$X(t) = c = X(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

which is the general solution to $x' = A(t)x$, are varied.

Definition 59 *Given the above inhomogeneous system and fundamental matrix, then*

$$c(t) = c(t_0) + \int_{t_0}^{t} X^{-1}(s)B(s)ds$$

and the general solution of the inhomogeneous equation has thus the form

$$X(t)\left(c(t_0) + \int_{t_0}^{t} X^{-1}(s)B(s)ds\right)$$

Moreover, to solve the IVP (with $x(t_0) = x_0$) one takes $X(t)$ to be such that $X(t_0) = E$ and sets $c_0 = x_0$.

9.3 Vector fields and their flows

Consider a system of $1-st$ order ODEs $x' = F(x)$. Where $F = (f_1 \dots f_n) : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k. The map F is also called a C^k **vector field** since it assigns to each point $x(x_1 \dots x_n)$ a vector $F(x) \in \mathbb{R}^n$.

Another represenation of a vector field is that of a map

$$V : U \to U \times \mathbb{R}^n, \quad V(x) = (x, F(x))$$

which assigns to every point $x \in U$ a vector $F(x) \in \mathbb{R}^n$ attached to x.

Remark: Geometrically a solution of the ODE $x' = F(x)$ is a curve $x = x(t)$ that is tangent to the vector field at every point and, moreover, the magnitude and direction of $x'(t)$ are equal to that of $F(x(t))$.

Remark: ODEs are sometimes written as $x' = V(x)$, where V is a vector field (or $x' = V(t,x)$ in time-dependent case).

Definition 60 *When $F = F(x)$ ($V = V(x)$) is independent of t, the ODE $x' = F(x)$ is called* ***autonomous***

Definition 61 *The flow of a (at least Lipschitz) vector field is a (locally defined) map*

$$g^t(x) : (-\epsilon, \epsilon) \times U \to \mathbb{R}^n$$

as follows: $g^t(x)$ is the unique (maximal) solution of $x' = F(x)$ with $g^0(x) = x$.

Proposition 62 *For an autonomous ODE $x' = F(x)$ and $t, s \in \mathbb{R}$, $|t| < \epsilon$, $|s| < \epsilon$, $|t+s| < \epsilon$, one has*

$$g^{t+s}(x) = g^t(g^s(x)) = g^s(g^t(x))$$

Corollary 63 *Assume that solutions of $x' = F(x)$ are defined for all $t \in \mathbb{R}$. Then the flow $g^t(x)$ defines a group homomorphism*

$$t \in \mathbb{R} \to g^t(\cdot)$$

from $\mathbb{R}$ into the group of maps from $U \subset \mathbb{R}^n$ to itself

Remark: if $F \in C^k$ then

$$g^t(x) : (-\epsilon, \epsilon) \times U \to U$$

is of class at least C^{k-1}

Theorem 64 *Consider a C^∞ vector field $V(x)$ on $U \subset \mathbb{R}^n$. Assume that all solutions of $x' = V(x)$ are defined for all times $t \in \mathbb{R}^n$. Then the flow g^t of V*

$$g^t(x) : \mathbb{R} \times U \to U$$

is C^∞ smooth. Moreover, for each fixed t_0, the map

$$g^{t_0}(\cdot) : U \to U$$

is a C^∞ diffeomorphism.

10 Multiple integrals

Definition 65 *Let $f : I^n = \prod_{i=1}^n [a_i, b_i] \to \mathbb{R}$ be a real-valued function. Consider a partition*

$$P_i : a_i = x_0^i < x_1^i < \cdots < x_{k_i}^i = b_i$$

of each segment $[a_i, b_i]$ and the resulting partition of the box I^n by smaller boxes $I^n_{i_1,\dots,i_n}$,

$$I^n_{i_1,\dots,i_n} = [x^1_{i_1}, x^1_{i_1+1}] \times \cdots \times [x^n_{i_n}, x^n_{i_n+1}]$$

Then, a Rieman Sum *is a sum of the form*

$$R(f, P, S) = \sum_{i_1=0}^{k_1-1} \cdots \sum_{i_n=0}^{k_n-1} f(S_{i_1,\dots,i_n}) \left| I^n_{i_1,\dots,i_n} \right|$$

where $S_{i_1,\dots,i_n} \in I^n_{i_1,\dots,i_n}$ are sample points and $\left| I^n_{i_1,\dots,i_n} \right|$ is the volume of $I^n_{i_1,\dots,i_n}$, i.e. the product o fthe lengths of its sides:

$$|x^1_{i_1+1} - x^1_{i_1}| \times \cdots \times |x^n_{i_n+1} - x^n_{i_n}|$$

Definition 66 *A function $f : I^n = \prod_{i=1}^n [a_i, b_i] \to \mathbb{R}$ is called Rieamann-integrable on I^n with integral*

$$J = \int \cdots \int_{I^n} f(x) dx^1 \cdots dx^n$$

if $\forall \epsilon > 0$ $\exists \delta > 0$ such that for all partitions P of I^n consisting of boxes $I^n_{i_1,\dots,i_n}$ with diameter $d(I^n_{i_1,\dots,i_n})$ less than δ and any choice of sample points $S_{i_1,\dots,i_n}$, $S_{i_1,\dots,i_n} \in I^n_{i_1,\dots,i_n}$, for the corresponding Riemann sum

$$|J - R(f, P, S)| < \epsilon$$

In other words, f is Riemann integrable when the following limit exists

$$J := \lim_{d(p) \to 0} R(f, P, S)$$

where the limit is taken along all marked partitions (P, S) with the diameter $d(p) := \max_{i_1,\dots,i_n} d(I^n_{i_1,\dots,i_n})$ tending to zero.

Proposition 67 *If $f : I^n \to \mathbb{R}$ is Riemann-integrable, then it is bounded.*

Definition 68 *Consider a function $f : I^n \to \mathbb{R}$ and let P be a partition of I^n. Set*

$$m_{i_1,\dots,i_n} := \underset{x\in I^n_{i_1,\dots,i_n}}{inf} \quad f(x) \quad M_{i_1,\dots,i_n} := \underset{x\in I^n_{i_1,\dots,i_n}}{sup} \quad f(x)$$

The sums

$$r(f,P) = \sum_{i_1=0}^{k_1-1} \cdots \sum_{i_n=0}^{k_n-1} m_{i_1,\dots,i_n} \left|I^n_{i_1,\dots,i_n}\right|$$
$$R(f,P) = \sum_{i_1=0}^{k_1-1} \cdots \sum_{i_n=0}^{k_n-1} M_{i_1,\dots,i_n} \left|I^n_{i_1,\dots,i_n}\right|$$

are called, respectively, Lower *and* Upper Darboux sums *of f relative to P*

Definition 69 *Consider a function $f : I^n \to \mathbb{R}$ and let P be a partition of I^n. Then, the quantaties*

$$\int_{-\, I^n} f dx^1 \cdots dx^n := \underset{P}{sup}\, r(f,P)$$
$$\bar{\int}_{I^n} f dx^1 \cdots dx^n := \underset{P}{inf}\, R(f,P)$$

are called, respectively, Lower *and* Upper integrals *of f on I^n.*

Remark: Note that the $\underset{P}{sup}(\underset{P}{inf})$ is taken with respect to all partitions P of I^n

Lemma 70 *For all marked partitions (P, S), we have*

- $r(f,P) \leq R(f,P,S) \leq R(f,P)$
- $r(f,P) \leq \int_{-I^n} f dx \leq \bar{\int}_{I^n} f dx \leq R(f,P)$

Theorem 71 (Darboux's criterion of Riemann integrability) *A function $f : I^n \to \mathbb{R}$ is Riemann integrable iff f is bounded and the lower and upper integrals coincide*

$$\bar{\int}_{I^n} f dx = \int_{-\, I^n} f dx$$

Remark: For a bounded function, the lower and upper integrals of f always exist. This follows from the above lemma ii.

Definition 72 (zero set) *A subset $Z \subset \mathbb{R}^n$ is a* zero set *(or of* Lebesgue measure zero*) if $\forall\epsilon > 0$ there exists a countable covering of Z by (open or equivalently closed) boxes I^n_j such that*

$$\sum_j \left|I^n_j\right| < \epsilon$$

Proposition 73 *Let $Z \subset \mathbb{R}^n$ be a zero set. Then $W \subset Z$ is also a zero set and a countable union of zero sets is again a zero set*

Theorem 74 (Riemann-Lebesgue theorem/Lebesgue's criterion) *A function $f : I^n \to \mathbb{R}$ is riemann integrable iff it is bounded and continuous almost everywhere on I^n,i.e., there exists a zero set $Z \subset I^n$ such that f is continuous on $I^n \backslash Z$*

Definition 75 *Let E be a bounded subset of $\mathbb{R}^n$ and χ_E be the indicator function (takes value 1 if $x \in E$ and 0 otherwise). A function $f : E \to \mathbb{R}$ is Riemann integrable on E if the function $f \cdot \chi_E(x)$ is Riemann integrable on some box I^n containing E. The integral of f over E is then defined by*

$$\int\limits_E f\, dx := \int\limits_{E \subset I^n} f \cdot \chi_E\, dx$$

Proposition 76 *If I_1^n and I_2^n contain E, then $\int\limits_{E \subset I_1^n} f \cdot \chi_E\, dx$ and $\int\limits_{E \subset I_2^n} f \cdot \chi_E\, dx$ either both exist and are equal or both do not exist.*

Theorem 77 *Let $E \subset \mathbb{R}^n$ be a bounded subset of $\mathbb{R}^n$ such that the boundary is a zero set. Then a function $f : E \to \mathbb{R}$ is Riemann integrable iff f is continuous almost everywhere (i.e. f continuous outside a zero set $Z \subset E \subset \mathbb{R}^n$).*

Moreover, if E is bounded and ∂E is not a zero set, then χ_E is not Riemann integrable on E.

Definition 78 *A volume of $E \subset \mathbb{R}^n$ (if exists) is the Riemann integral*

$$\int_E 1 \cdot dx^1 \cdots dx^n$$

Theorem 79 (Change of variables formula) *Let $\psi : U \to W$ be a C^1-diffeomorphism between subsets U and W of $\mathbb{R}^n$. Let E be a bounded subset of $\mathbb{R}^n$ such that $\overline{E} \subset W$. If f is Riemann integrable on E, then $g := (f \circ \psi) \cdot |det(D\psi)|$ is Riemann integrable on $\psi^{-1}(E)$ and*

$$\int\limits_{\psi^{-1}(E)} (f \circ \psi) \cdot |det(D\psi)|\, dx = \int\limits_E f\, dy, \quad y = \psi(x)$$

Theorem 80 (Fubini's theorem) *Assume that $E_1 \subset \mathbb{R}^k$ and $E_2 \subset \mathbb{R}^m$ are bounded and let $f : E_1 \times E_2 \to \mathbb{R}$ be Riemann integrable. Then both $\int\limits_{-E_1} f(x, y)dx$ and $\bar{\int}_{E_1} f(x, y)dx$ exists and integral on E_2 with respect to y, then*

$$\int_{E_2} \left(\int\limits_{-E_1} f(x, y)dx \right) dy = \int_{E_2} \left(\bar{\int}_{E_1} f(x, y)dx \right) dy = \int\int\limits_{E_1 \times E_2} f(x, y)dxdy$$

where $x = (x^1, ..., x^k)$ and $y = (x^{k+1}, ..., x^{k+m})$.

Corollary 81 *Assume that $E_1 \subset \mathbb{R}^k$ and $E_2 \subset \mathbb{R}^m$ are bounded and let $f : E_1 \times E_2 \to \mathbb{R}$ be Riemann integrable. Then*

$$\int\int_{E_1 \times E_2} f(x,y)dxdy = \int_{E_2} \left(\underline{\int}_{E_1} f(x,y)dx \right) dy = \int_{E_1} \left(\underline{\int}_{E_2} f(x,y)dy \right) dx$$

Corollary 82 *Under preceding assumption $\int_{E_1} f(x,y)dx$ exists for almost all y. Similarly, $\int_{E_2} f(x,y)dy$ exists for almost all x*

Corollary 83 *If $f : I_1^k \times I_2^m \to \mathbb{R}$ and f is continuous, then the iterated integrals exist and are equal to each other.*

Corollary 84 *Assume $D \subset \mathbb{R}^{n-1}$ bounded, let $\psi_1, \psi_2 : D \to \mathbb{R}$ and*

$$E = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, x \in D \, \psi_1(x) \leq y \leq \psi_2(x)\}$$

and is bounded. If $f : E \to \mathbb{R}$ is integrable then

$$\int_E f(x,y)dxdy = \int_D \int_{\psi_1}^{\psi_2} f(x,y)dydx$$

Corollary 85 (Cavalieri's principle) *Let $E \subset \mathbb{R}^{n-1}$ be bounded and let ∂E be a zero set. Then*

$$Vol(E) = \int_E dx^1 \cdots dx^{n-1}dy = \int_{I_y^1} Vol(E_y)dy$$

where $E \subset I_x^{n-1} \times I_y^1$, $E_{y_0} = \{(x,y) \in E | y = y_0\}$ is a y-slice of E and $Vol(E_{y_0})$ is its $(n-1)-$volume; more precisely, any number between $\underline{\int}_{E_{y_0}} 1 \cdot dx$ and $\overline{\int}_{E_{y_0}} 1 \cdot dx$

Remark: Note that by corollary, $\int_{I_x^{n-1}} \chi_{E_y} dx$ exists almost everywhere, so $Vol(E_y)$ is well defined almost everywhere and also ∂E_y is a zero set in $\mathbb{R}^{n-1}$ for almost all y

Remark: The notion of the volume: $Vol(E) = \int_E 1 \cdot dx$, as follows from the Riemann Lebesgue theorem, is well defined precisely for those bounded sets E, for which ∂E is a zero set. Note also that it is invariant under Euclidean isometries by the change of variables formula

10.1 Improper Integrals

Definition 86 *Let $E = \bigcup_{j=1}^{\infty} E_j$, where $E_j \subset E_{j+1} \subset \mathbb{R}^n$, each E_j is bounded and for each j, ∂E_j is a zero set. Assume that $f : E \to \mathbb{R}$ is integrable on E_j for all j and*

$$J = \lim_{j \to \infty} \int_{E_j} f \, dx$$

exists and doesn't depend on E_j, then J is the improper integral *of f on E (the same notation $\int_E f \, dx = J$ is used)*

Proposition 87 *If $E = \bigcup_{j=1}^{\infty} E_j$, where $E_j \subset E_{j+1} \subset \mathbb{R}^n$, each E_j is bounded and for each j, ∂E_j is a zero set, and E is also bounded with ∂E a zero set, then*

- $\lim_{j\to\infty} Vol(E_j) = Vol(E)$
- *For every integrable function $f : E \to \mathbb{R}$, its restriction to E_j is also integrable and*

$$\lim_{j\to\infty} \int_{E_j} f\, dx = \int_E f\, dx$$

Remark: This proposition shows that improper Riemann integrals generalise Riemann integrals

Proposition 88 *If $f, g : E \to \mathbb{R}$ are both integrable on E_j (E_j bounded and ∂E_j a zero set), $|f| \leq g$ on E and $\lim_{j\to\infty} \int_{E_j} g\, dx$ exists, then*

$$\int_E g\, dx, \quad \int_E |f|\, dx, \quad \int_E f\, dx$$

exist

Definition 89 *Assume that for all $y \in [c, d) \subset \mathbb{R}$, the following improper integral exists:*

$$F(y) = \int_a^b f(x, y) dx$$

where $[a, b) \subset \mathbb{R}$ and b is possibly $+\infty$. It is assumed that on each segment $[a, c] \subset [a, b)$, a proper Riemann integral exists.

The improper integral converges uniformly *on $[e, d)$ if $\forall \epsilon > 0$ there exist a neighbourhood of the form (b_0, b) (or $(b_0, +\infty)$ where $b = +\infty$) such that $\forall c$ in this neighbourhood and $\forall y \in [e, d)$*

$$\left| \int_c^b f(x, y) dx \right| < \epsilon$$

Theorem 90 *Assume that $f = f(x, y)$ and $g = g(x, y)$, defined on $[a, b) \times [c, d)$ are integrable w.r.t x on all $[a, e) \subset [a, b)$ for all $y \in [c, d)$. If $|f(x, y)| \leq g(x, y)$ and $\int_a^b g(x, y)dx$ converges uniformly on $[c, d)$, then so does the integral $\int_a^b f(x, y)dy$ (in particular, it is well defined $\forall y \in [c, d)$)*

Theorem 91 *If $f : [a, b) \times [c, d)$ is continuous, the integrals*

$$\int_a^b f(x, y)dx \quad \text{and} \quad \int_c^d f(x, y)dy$$

converge uniformly w.r.t y on all $[c, e) \subset [c, d)$ and w.r.t x on all $[a, r) \subset [a, b)$, respectively, and there exists at least one iterated integral

$$\int_c^d \int_a^b |f|\, dx\, dy \quad \text{or} \quad \int_a^b \int_c^d |f|\, dy\, dx$$

then

$$\int_c^d \int_a^b f\, dx\, dy = \int_a^b \int_c^d f\, dy\, dx$$

Theorem 92 *If $f = f(x,y) : [a,b) \times [c,d] \to \mathbb{R}$ and its partial derivative with respect to y are continuous, the integral*

$$\int_a^b f'_y(x,y)dx$$

converges uniformly on $[c,d]$ and $\int_a^b f(x,y)dx$ converges for at least one y in $[c,d]$, then $\int_a^b f(x,y)dy$ converges uniformly and

$$\frac{\partial}{\partial y}\int_a^b f\,dx = \int_a^b f'_y\,dx$$

Corollary 93 *Let $f : [a,b] \times [c,d] \to \mathbb{R}$. Assume $f \in C^0$ and $\frac{\partial f}{\partial y}$ exists and is C^0. Then*

$$\int_a^b f(x,y)dx \in C^1(y)$$

and

$$\frac{\partial}{\partial y}\int_a^b f(x,y)dx = \int_a^b \frac{\partial}{\partial y}f(x,y)dx$$

11 Alternating k-linear forms

Definition 94 *Given a vector space V over a field $\mathbb{K}$, its dual $V^\star$ is defied as*

$$V^\star = \mathcal{L}(V,\mathbb{K})$$

the space of linear functions from V to $\mathbb{K}$

Remark: $V^\star$ is itself a vector space over $\mathbb{K}$.

Definition 95 *Let V be finite-dimensional (and hence isomorphic to $\mathbb{K}^n$, $n < \infty$) and $e_1,...,e_n \in V$ be a basis of V. The* ***dual basis*** *of $e_1,...,e_n$ is the basis*

$$e^1,...,e^n \in V^\star$$

of $V^\star$ defined by the rule $e^i(e_j) = \delta^i_j \ \forall i,j \le n$

Remark: δ^i_j is known to be the Kronecker delta and is equal to 1 when $j = i$ and 0 otherwise.

Proposition 96 *Let $e_1,...,e_n$ be a basis of V. Then the dual basis $e^1,...,e^n$ is indeed a basis of $V^\star$. Moreover,*

1. $v = \sum_{i=1}^n v^i e_i \Rightarrow v = \sum_{i=1}^n e^i(v)e_i$
2. *for any $f \in V^\star$, $f = \sum_{i=1}^n f(e_i)e^i$*

Definition 97 *Elements of $V^\star = \mathcal{L}(V,\mathbb{K})$ are called* ***Linear functions*** *or* ***linear 1-forms*** *or* ***covectors***

Definition 98 *Given two vectors spaces V_1 and V_2 over a field $\mathbb{K}$, a function*

$$\omega : V_1 \times V_2 \to \mathbb{K}$$

*is called **bilinear** or a **bilinear form** if it is linear in each argument, that is, if*

$$\omega(\lambda v + u, s) = \lambda\omega(v, s) + \omega(u, s)$$
$$\omega(v, \lambda s + r) = \omega(v, s) + \lambda\omega(v, r)$$

*for all $v, u \in V_1$; $s, r \in V_2$, and $\lambda \in \mathbb{K}$. The **vector** space of all bilinear maps $w : V_1 \times V_2 \to \mathbb{K}$ is denoted by $\mathcal{L}(V_1 \times V_2, \mathbb{K})$*

Remark: Recall that the space $\mathcal{L}(V_1 \times V_2, \mathbb{K})$ is isomorphic to the space $\mathcal{L}(V_1\mathcal{L}(V_2, \mathbb{K}))$.

Definition 99 *Given vectors spaces $V_1, ..., V_k$ over $\mathbb{K}$, a function $\omega : V_1 \times \cdots \times V_k \to \mathbb{K}$ is called **k-linear** or a k-linear form if it is linear in each argument, i.e., if $\forall i$, $1 \leq i \leq k$ and $\forall v_j \in V_j$ $j \neq i$*

$$\omega(v_1, ..., v_{i-1}, \cdot, v_{i+1}, ..., v_k) : V_i \to \mathbb{K}$$

is linear. The space of k-linear maps is denoted by $\mathcal{L}(V_1\times\cdots\times V_k, \mathbb{K}) \equiv \mathcal{L}(V_1, \mathcal{L}(V_2, \cdots \mathcal{L}(V_k, \mathbb{K}) \cdots))$

remark: More generally, one can consider k-linear maps with values in another vector space, rather than the field $\mathbb{K}$.

Definition 100 *Let V be a real vector space. A k-linear map $\omega : V \times \cdots \times V \to \mathbb{R}$ is called **alternating** if for every permutation σ of $\{1, ..., k\}$ and every choice of $v_1, ..., v_k \in V$, we have*

$$\omega(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = sign(\sigma)\omega(v_1, ..., v_k)$$

The space of k-linear alternating maps $\omega : V \times \cdots \times V \to \mathbb{R}$ is denoted by $\bigwedge^k(V)^\star$.

Theorem 101 *The space $\bigwedge^k(V)^\star$ is a vector space*

11.1 Wedge product

Definition 102 *Let V be a real vector space and α, β be two linear functions on V (i.e. elements of $V^\star = \mathcal{L}(V, \mathbb{R})$). The wedge product of α and β is the map*

$$\alpha \wedge \beta : V \times V \to \mathbb{R}$$

defined by

$$\alpha \wedge \beta(v_1, v_2) = \det \begin{bmatrix} \alpha(v_1) & \alpha(v_2) \\ \beta(v_1) & \beta(v_2) \end{bmatrix} = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

Proposition 103 *The wedge product of $\alpha \wedge \beta$ of two linear functions $\alpha, \beta \in V^\star = \bigwedge^1(V)^\star$ is an alternating bilinear form, i.e. an element of $\bigwedge^2(V)^\star$*

Definition 104 *Let V be a real vector space and $\alpha_1, ..., \alpha_k$ be in $V^\star = \bigwedge^1(V)^\star$. The wedge product of $\alpha_1, ..., \alpha_k$ is the map*

$$\alpha_1 \wedge \cdots \wedge \alpha_k : V \times \cdots \times V \to \mathbb{R}$$

defined by

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, ..., v_k) = \det \begin{bmatrix} \alpha_1(v_1) & \cdots & \alpha_1(v_k) \\ \vdots & & \vdots \\ \alpha_k(v_1) & \cdots & \alpha_k(v_k) \end{bmatrix}$$

where $v_1, ..., v_k$ are arbitrary vectors in V

Proposition 105 *The wedge product $\alpha_1 \wedge \cdots \wedge \alpha_k$ of k linear functions $\alpha_1, \cdots, \alpha_k \in V^\star = \bigwedge^1(V)^\star$ is an alternating k linear form,i.e., an element of $\bigwedge^k(V)^\star$*

Theorem 106 *Let V be an n-dimensional real vector space and $e_1, ..., e_n$ be a basis of V. Then wedge products*

$$e^{j_1} \wedge \cdots \wedge e^{j_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n$$

form a basis of $\bigwedge^k(V)^\star$ for $1 \leq k \leq n$. In particular, $\dim(\bigwedge^k(V)^\star = C^n_k$ (binomial coefficient) for $1 \leq k \leq n$. For $k > n$, $\bigwedge^k(V)^\star$ has dimension zero

Definition 107 *Let V be an n-dimensional real vector space and $e_1, ..., e_n$ basis of V. The wedge product of $\omega \in \bigwedge^k(V)^\star$ and $\eta \in \bigwedge^l(V)^\star$ is defined by*

$$\omega \wedge \eta = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_l}} \omega_{i_1, ..., i_k} \eta_{j_1, ..., j_l} e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_l}$$

where

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, ..., i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \quad \eta = \sum_{j_1 < \cdots < j_l} \eta_{i_1, ..., i_k} e^{j_1} \wedge \cdots \wedge e^{j_l}$$

Remark: In other words, one defines the wedge product of basic k-forms $e^{i_1} \wedge \cdots \wedge e^{i_k} \in \bigwedge^k(V)^\star$ and $e^{j_1} \wedge \cdots \wedge e^{j_l} \in \bigwedge^l(V)^\star$ as

$$e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_l} \in \bigwedge^{k+l}(V)^\star$$

and then extends this definition to arbitrary $\omega \in \bigwedge^k(V)^\star$ and $\eta \in \bigwedge^l(V)^\star$

Proposition 108 *This defied wedge product is consistent with the wedge product of Linear functions defined above and is independent of the choice of the basis $e_1, ..., e_n \in V \cong \mathbb{R}^n$*

Definition 109 *Let V be a real vector space. The wedge product $\omega \wedge \eta$ of two alternating forms $\omega \in \bigwedge^k(V)^\star$ and $\eta \in \bigwedge^l(V)^\star$ is defined by*

$$\omega \wedge \eta(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_\sigma sign(\sigma) \omega(v_1, ..., v_k) \eta(v_{k+1}, ..., v_{k+l})$$

where $\sigma = (i_1, ..., i_{k+l}$ is a permutation of $\{1, ..., k+l\}$ such that $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_{k+l}$

Corollary 110 *Let $\omega \in \bigwedge^k(V)^\star$, $\eta \in \bigwedge^l(V)^\star$ and $\phi \in \bigwedge^m(V)^\star$. Then*

- $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$
- $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$
- *The wedge product is linear in each of its arguments, e.g.:*

$$\omega \wedge (c\eta) = c(\omega \wedge \eta), \quad \text{for } c \in \mathbb{R}$$
$$\omega \wedge (\eta + \phi) = \omega \wedge \eta + \omega \wedge \phi, \quad \text{in case } l = m$$

11.2 Pull-Back

Definition 111 *Let $f : V \to W$ be a linear map between real vector spaces V and W. The* ***pull-back*** *of an alternating k-linear form $\omega \in \bigwedge^k(W)^\star$ is the alternating k-linear form $f^\star\omega \in \bigwedge^k(V)^\star$ defined by the rule*

$$f^\star\omega(v_1, ..., v_k) = \omega(f(v_1), ..., f(v_k))$$

where $v_1, ..., v_k$ are arbitrary vectors in V

Proposition 112 *Let $f : V \to W$ be a linear map between real vector spaces and let $\omega \in \bigwedge^k(W)^\star$,$\eta \in \bigwedge^l(W)^\star$. Then*

- *$f^\star\omega$ is an alternating k-linear form on V*
- *$f^s tar : \bigwedge^k(W)^\star \to \bigwedge^k(V)^\star$ is linear*
- *$f^s tar(\omega \wedge \eta) = f^\star\omega \wedge f^\star\eta$*

Theorem 113 *Let $f : U \to V$ and $g : V \to W$ be linear maps. Then $(g \circ f)^\star = f^\star \circ g^\star : \bigwedge^k(W)^\star \to \bigwedge^k(U)^\star$*

Differential forms

Definition 114 *Let U be an open subset on $\mathbb{R}^n$. A function*

$$\omega : U \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ factors}} \to \mathbb{R}, \quad \omega = \omega(x, v), x \in U, v \in \mathbb{R}^n$$

is called a ***differential*** *k form if for all fixed $x \in U$, the function $\omega(x, \cdot) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is k-linear and alternating.*

A differential k-form ω is C^m-smooth ($m \leq \infty$) when it is C^m smooth as a function $\omega : U \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$.

Proposition 115 *Let $\omega : U \times \mathbb{R}^{nk} \to \mathbb{R}$ be a differential k-form on U. Then there are unique functions*

$$\omega_{i_1 \dots i_k} : U \to \mathbb{R}, \quad i_1 < \cdots < i_k$$

such that

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k}(x) Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}$$

The k-form ω is C^m iff all functions $\omega_{i_1 \dots i_k}$ are C^m

Remark: Differential forms ω naturally act on vector fields.

Definition 116 *A differential k-form ω of class C^m on $U \subset \mathbb{R}^n$ is an alternating k-linear over $C^m(U)$ map*

$$\omega : \mathcal{X}^m(U) \times \cdots \times \mathcal{X}^m(U) \to C^m(U)$$

where $\mathcal{X}^m(U)$ is the space of all C^m-vector fields on U and $C^m(U)$ is the space of all C^m-smooth functions on U

Proposition 117 *Let $U \subset \mathbb{R}^n$ be open. If $\omega : U \times \mathbb{R}^{nk} \to \mathbb{R}$ is a differential k-form (in the sense of the first definition) of class C^m, then it induces an alternating k-linear over $C^m(U)$ map*

$$\tilde{\omega} : \mathcal{X}^m(U) \times \cdots \times \mathcal{X}^m(U) \to C^m(U)$$

by setting $\tilde{\omega}(v_1, ..., v_k)(x) = \omega(x, F_1(x), ..., F_k(x))$ where $v_i(x) = (x, F_i(x))$. Conversely, every such map $\tilde{\omega}$ (differential k-form in the sense of the second definition) induces a C^m function

$$\omega : U \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$$

such that $\forall x \in U$, $\omega(x, \cdot)$ is k-linear and alternating, by setting $\omega(x_0, c_1, ..., c_k) = \tilde{\omega}(v_1, ..., v_k)(x_0)$, where the vector fields $v_i(x) = (x, F(x))$ are C^m on U and such that $F_i(x) = c_i$ in a small neighbourhood of $x_0 \in U$ (here $c_i \in \mathbb{R}^n$ are constant vectors)

11.3 Operations on differential forms

Definition 118 *Let ω be a differential k-form on $U_2 \subset \mathbb{R}^m$ and let $f : U_1 \subset \mathbb{R}^n \to U_2 \subset \mathbb{R}^m$ be a C^1 map. The **pull-back** of ω under f is the differential k-form $f^\star\omega$ on U_1 defined by*

$$f^\star\omega(x, c_1, ..., c_k) = \omega(f(x), Df|_x(c_1), ..., Df|_x(c_k))$$

meaning that at each x, we have the pull-back of alternating k-linear forms under linear map df between the corresponding tangent spaces

Proposition 119 *Let $f : U_1 \subset \mathbb{R}^n \to U_2 \subset \mathbb{R}^m$ be of class (at least) C^1. The pull-back $f^\star$ is a linear map. Moreover, it respects the wedge product: if ω and η are differential k-forms on U_2, then $f^\star(\omega \wedge \eta) = f^\star\omega \wedge f^\star\eta$*

Proposition 120 *If $f : U_1 \subset \mathbb{R}^n \to U_2 \subset \mathbb{R}^m$ and $g : U_2 \to U_3 \subset \mathbb{R}^e$ are C^1 maps, then $(g \circ f)^\star = f^\star \circ g^\star$*

11.3.1 Exterior derivative

Definition 121 *Let ω be a differential k-form on $U \subset \mathbb{R}^n$ of class C^m, $m \geq 1$. The **exterior derivative** $D^{ext}\omega$ of ω is the differential $(k+1)$-form on U defined by*

$$D^{ext}\omega = \sum_{i_1 < ... < i_k} D\omega_{i_1 ... i_k}(x) \wedge Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}$$

where

$$\omega = \sum_{i_1 < ... < i_k} \omega_{i_1 ... i_k}(x) Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}$$

is the unique expansion of ω with respect to the basis k-forms $Dx^{i_1} \wedge \cdots \wedge Dx^{i_k}$, $i_1 < ... < i_k$

Proposition 122 *The following properties of the exterior derivative D^{ext} hold:*

- *$\forall$ differential k-forms ω and η of class C^1 and $\forall \lambda \in \mathbb{R}$, we have*

$$D^{ext}(\lambda\omega + \eta) = \lambda D^{ext}\omega + D^{ext}\eta$$

  i.e. D^{ext} is linear

- *$\forall$ diff. form ω and $\forall$ function f of class C^1,*

$$D^{ext}(f\omega) = Df \wedge \omega + fD^{ext}\omega$$

- *$\forall$ diff. form ω and $\forall$ map F of class C^2, we have*

$$F^\star(D^{ext}\omega) = D^{ext}(F^\star\omega)$$

- *$\forall$ diff. form ω of class C^2, $D^{ext}(D^{ext}\omega) = 0$*

Remark: Note that for differential 0-forms of class C^1, we have $D^{ext}f = Df$. Therefore, D^{ext} is an extension of the usual notion of the derivative to differential forms. Also note that unlike in the case of higher order derivatives $D^r f$ which might be non-zero for all $r \in \mathbb{N}$ we have that $D^{ext} \circ D^{ext} \equiv 0$, because of skew-symmetry of differential forms.

Note: In what follows, when considering differential forms we will use d to refer to both exterior derivative and 1-st differentials of functions. When talking about higher derivatives of maps, D will be used instead.

Definition 123 *A C^1 differential k-form ω on $U \subset \mathbb{R}^n$ is called* ***closed*** *if $d\omega \equiv 0$ on U.*

A C^1 differential k-form ω on $U \subset \mathbb{R}^n$ is called ***exact*** *if there exists a $k-1$ form η on U such that $\omega = d\eta$*

Proposition 124 *Every exact differential k-form ω on $U \subset \mathbb{R}^n$ is closed*

12 Vector Fields, differentials forms and the classical operations

Definition 125 *Vector Fields A vector field on $U \subset \mathbb{R}^n$ is a map*

$$v : U \to U \times \mathbb{R}^n, \quad v(x) = (x, F(x))$$

that assigns to each $x \in U$ a vector $F(x) \in \mathbb{R}^n$ "at x" (here $F : U \to \mathbb{R}^n$ is some map). A vector field v is C^m on U when $v = (id, F) : U \to U \times \mathbb{R}^n$ (or equivalently $F : U \to \mathbb{R}^n$) is C^m

Definition 126 *A vector field on $\mathbb{R}^n$ is a map $X : \mathbb{R}^n \to T\mathbb{R}^n$, where $T\mathbb{R}^n$ is the tangent bundle of $\mathbb{R}^n$ ($U \subset \mathbb{R}^n \times \mathbb{R}^n$), such that $\pi \circ X = id_{\mathbb{R}^n}$. In other words, X is of the form $X(p) = (p, v(p))$. The vector field is C^m if the function v is C^m.*

Proposition 127 *If v is a (C^m-smooth) vector field, then $w = \langle v, \cdot \rangle$ is a (C^m-smooth) one-form*

Definition 128 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on U. The vector field v on U such that*

$$\langle v, \cdot \rangle = df$$

*is called the **gradient vector field of** f and is denoted by* $\operatorname{grad} f$

Definition 129 *Let $v : U \subset \mathbb{R}^n \to U \times \mathbb{R}^n$ be a differentiable vector field on U. The **divergence** of v is the function $\operatorname{div}(v) : U \to \mathbb{R}$ defined by*

$$\operatorname{div}(v)(x) = \operatorname{trace} DF|_x$$

where $v = (x, F(x))$ and DF is expressed in Euclidean coordinates. Note that writing

$$F = (f_1(x^1, ..., x^n), ..., f_n(x^1, ..., x^n))$$

we have $\operatorname{div}(v) = \sum_{i=1}^n \frac{\partial f_i}{\partial x^i}$

Definition 130 *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. The **Laplacian** $\Delta f : U \to \mathbb{R}$ is a function on U given by*

$$\Delta f = \operatorname{div}(\operatorname{grad} f)$$

In Euclidean coordinates, $\Delta f = \sum_{i=1}^n \frac{\partial^2 f_i}{\partial x^{i2}}$

Definition 131 (Hodge star) *Given a differential k-form $\omega \in \Omega^k(U)$, $U \subset \mathbb{R}^n$, its Hodge star is the $(n-k)$-form $\star\omega \in \Omega^{n-k}(U)$ defined by extending the assignment*

$$\star dx^{i_1} \wedge \cdots \wedge dx^{i_k} = (-1)^\sigma dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}$$

where $i_1 < \cdots < i_k$,$j_1 < \cdots < j_{n-k}$ and $\sigma(i_1, ..., i_k, j_1, ..., j_{n-k})$ is the permutation of $\{1, ..., n\}$ by sky-linearity.

Remark: Note that for a k-form ω on $U \subset \mathbb{R}^n$, $\star\star\,\omega = (-1)^{k(n-k)}\omega$

Proposition 132 *Let v be a differentiable vector field on $U \subset \mathbb{R}^n$. Then the divergence of v satisfies*

$$\operatorname{div}(v) = \star(d \star \omega), \quad \text{where } \omega = \langle v, \cdot \rangle$$

Proposition 133 *Let $f : U \subset \mathbb{R}^n$ be a C^2 function. Then*

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \star d \star f$$

Definition 134 (Rotational) *Let v be a differentiable vector field on $U \subset \mathbb{R}^n$. The rotational of v is the $(n-2)$-form $\operatorname{rot}(v)$ defined by*

$$\operatorname{rot}(v) = \star(d\omega), \quad \text{where } \omega = \langle v, \cdot \rangle$$

If $n = 3$, then $\operatorname{rot}(v)$ is a $(3-2=1)$-form, and hence there is a vector field, called the curl *of v and denoted by $\operatorname{curl}(v)$ such that*

$$\langle \operatorname{curl}(v), \cdot \rangle = \operatorname{rot}(v) = \star(d\omega)$$

Proposition 135 *Consider* $\mathbb{R}^3$ *with Euclidean coordinates* x, y, z *and let* $v = (id, F) : U \subset \mathbb{R}^3 \to U \times \mathbb{R}^3$*, where* $F = (f_1, f_2, f_3)$*, be a differentiable vector field on* U*. Then*

$$\operatorname{curl}(v) = \nabla \times F = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

Remark: Let $\alpha_1 = \langle v_1, \cdot \rangle$ and $\alpha_2 = \langle v_2, \cdot \rangle$ be linear functions on $\mathbb{R}^3$ with standard inner product then $\langle v_1 \times v_2, \cdot \rangle = \star(\alpha_1 \wedge \alpha_2)$

Proposition 136 *Let* $f : U \subset \mathbb{R}^n \to \mathbb{R}$ *be a* C^2 *function and* $v : U \to U \times \mathbb{R}^n$ *be a* C^2 *vector field. Then*

- $\operatorname{rot}(\operatorname{grad} f) = 0$
- $\operatorname{curl}(\operatorname{grad} f) = 0$ *($n = 3$)*
- $\operatorname{div}(\operatorname{curl} v) = 0$ *($n = 3$)*

13 Integration of differential forms

Definition 137 *A subset* $M_k \subset \mathbb{R}^n$ *is a regular* C^∞*-smooth* k*-dimensional surface in* $\mathbb{R}^n$ *if for every point* $x \in M_k$ *there exists an open neighbourhood* $x \in U$ *in* $\mathbb{R}^n$ *such that*

$$M_k \cap U = \{(z, y) \in U \,|\, y = F(z)\}$$

in a graph of a smooth map $F : V \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$*, where* $z = (x^{i_1}, ..., x^{i_k})$*,* $y = (x^{j_1}, ..., x^{j_{n-k}})$ *with* $i_1 ... i_k, j_1 ... j_{n-k}$ *all distinct*

Definition 138 *A regular level set* $M_c = \{f = c\}$ *of a smooth function* $f : \mathbb{R}^n \to \mathbb{R}$*,i.e., such that* $\operatorname{grad} f(x) \neq 0$ *for every* $x \in M_c$*, is a regular* C^∞*-smooth* $(n-1)$*-dimensional surface in* $\mathbb{R}^n$*, whenever* M_c *is non-empty. More generally,*

$$M_{c_1,...,c_{n-k}} = \{x \in \mathbb{R}^n | F(x) = (c_1, ..., c_{n-k})\} \neq \emptyset$$

of a C^∞*-smooth map* $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$*, i.e., such that rank* $DF|_x = n - k \ \forall x \in M_{c_1,...,c_{n-k}}$*, is a regular* C^∞*-smooth* k*-dim surface in* $\mathbb{R}^n$*.*

Definition 139 *Let* M^k *be a regular* C^∞*-smooth surface in* $\mathbb{R}^n$*. A differential* k*-form on* M^k *is a field of alternating* k*-linear functions*

$$\omega|_x : T_x M^k \times \cdots \times T_x M^k \to \mathbb{R}$$

where $T_x M^k = \{v \in \mathbb{R}^n | v = \dot{\gamma}(0)$*, with* γ *a* C^1 *differential curve in* $M^k \subset \mathbb{R}^n$ *with* $\gamma(0) = x\}$ *is the tangent space of* M^k *at* x

Definition 140 *A smooth regular* k*-dim. surface* M_k *is called orientable if it admits a smooth nowhere vanishing top (=of degree* k*) form*

Theorem 141 *A compact regular C^∞ smooth k-surface M^k in $\mathbb{R}^n$ is a finite union of $F_i(D_i)$ where*

$$F_i : \mathbb{R}^k \to M^k \subset \mathbb{R}^n, \quad F_i \in C^\infty(\mathbb{R}^k)$$

is such that F_i is a C^∞ diffeomorphism onto its image in M^k, $D_i \subset \mathbb{R}^n$ is a k-dim compact convex polyhedron and $F_i(int\, D_i) \cup F_j(int\, D_j) =, i \neq j$

Definition 142 *Any C^1 differentiable map $F : D \subset \mathbb{R}^k \to \mathbb{R}^n$ where D is a compact convex k-dim. polyhedron in $\mathbb{R}^k$, together with an orientation $\pm$ on D, is called k-dimensional cell. If $\sigma = (F, \pm D)$ is a k-dim cell and ω is a k-form on $\mathbb{R}^n$, then*

$$\int_\sigma \omega = \int_D F^\star \omega = \pm \int_D g(x) dx^1 ... dx^k$$

where $F^\star \omega = g(x) dx^1 ... dx^k$

Definition 143 *A k-chain in $\mathbb{R}^n$ (or a regular smooth surface $M^e \subset \mathbb{R}^n$) is a finite formal sum of the form*

$$c_k = \sum_{i=1}^n m_i \sigma_i$$

where $m_i \in \mathbb{Z}$ and $\sigma_i = (F_i, \pm D_i)$ is a k-cell, take up to the natural equivalence relation:

$$m_1 \sigma + m_2 \sigma = (m_1 + m_2)\sigma, \quad -\sigma = (F, -D)$$

The set of chains because an abelian group under the formal sum operation and we define the integral of a differential k-form ω over a k-chain C_k as

$$\int_{C_k} \omega = \sum_{i=1}^n m_i \int_{\sigma_i} \omega$$

where $C_k = \sum_{i=1}^n m_i \sigma_i$

Proposition 144 *Let D_1 and D_2 be two compact convex polyhedral in $\mathbb{R}^k$ and let*

$$F : U \to V, \quad D_1 \subset U \subset \mathbb{R}^k, D_2 \subset V \subset \mathbb{R}^k$$

be a C^1 smooth diffeomorphism sending D_1 onto D_2 and preserving the orientation on $\mathbb{R}^k$. Then for any (C^0) k-form ω on D_2

$$\int_{D_1} F^\star \omega = \int_{D_2} \omega$$

Corollary 145 *Let $\sigma = (f, D)$ be a k-cell and ω a differential k-form in $\mathbb{R}^n$. If $F : \mathbb{R}^k \to \mathbb{R}^k$ is a C^1 diffeomorphism, then*

$$\int_{F^{-1}(\sigma)} F^\star \omega = \int_\sigma \omega$$

where $F^{-1}(\sigma) = (f \circ F, F^{-1}(D))$. Similarly, if $C_k = \sum_{i=1}^n m_i \sigma_i$ is a k-chain, then

$$\int_{F^{-1}(C_k)} F^\star \omega = \int_{C_k} \omega$$

where $F^{-1}(C_k) = \sum_{i=1}^k m_i F^{-1}(\sigma_i)$

13.1 Stokes's theorem

Definition 146 *Let $\sigma(F, D)$, $F : D \subset \mathbb{R}^k \to \mathbb{R}^n$ be a k-cell. The boundary $\partial\sigma$ is defined by*

$$\partial\sigma = \sum_i \sigma_i^{k-1}, \quad \sigma_i^{k-1} = (F|_{D_i}, D_i)$$

where D_i are the faces of D oriented by the outward normal $\vec{n}$. The boundary ∂C_k of a k-chain is defined by

$$\partial C_k = \sum_{j=1}^{n} m_j \partial\sigma_j$$

Proposition 147 *For a k-chain C_k, $\partial\partial C_k = 0$*

Theorem 148 (Stokes's theorem) *Let ω be a C^1-smooth differential $(k-1)$-form on $\mathbb{R}^n$ (or on a compact regular orientable C^∞-smooth surface M in $\mathbb{R}^n$). Then for every k-chain C_k in $\mathbb{R}^n$ (contained in M)*

$$\int_{C_k} d\omega = \int_{\partial C_k} \omega$$

Corollary 149 *Let M be a compact regular orientable C^∞-smooth k-surface in $\mathbb{R}^n$ with boundary ∂M. Then for every C^1-differentiable $(k-1)$-form ω on M,*

$$\int_M d\omega = \int_{\partial M} g^*(\omega)$$

where $g : \partial M \to M$ denotes the inclusion map.

Definition 150 *A subset $M^k \subset \mathbb{R}^n$ is a regular C^∞-smooth k-dimensional surface with boundary if for every point $x \in M^k$ there exists an open neighbourhood U of x in $\mathbb{R}^n$ such that*

$$M^k \cap U = \{(z, y) \in U | y = F(z)\}$$

is a graph of a smooth map $F : W \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$, where $z = (x^{i_1}, ..., x^{i_k})$, $y = (x^{j_1}, ..., x^{j_{n-k}})$ with $i_1, ..., i_k, j_1, ..., j_{n-k}$ all distinct and W is either

- *an open ball $B_r(z_0)$ or*
- *a part of $B_r(z_0)$ cut out by a C^∞ function $f : W = B_r(z_0) \cap \{f \leq 1\}$*

assuming $\{f = 1\}$ is a regular level set for f on $\mathbb{R}^k$

Corollary 151 (Green's theorem) *Let $U \subset \mathbb{R}^2$ be an open bounded subset in $\mathbb{R}^2$ with ∂U a closed regular C^∞-smooth curve. If ω is a C^1-smooth $1-$form on (a neighbourhood of) $\overline{U}$, then*

$$\int_{M=\overline{U}} d\omega = \int_{C=\partial\overline{U}} \omega$$

which in coordinates (x, y) on $\mathbb{R}^2$ reads as

$$\int_M \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy = \int_C a(x, y)dx + b(x, y)dy$$

where $\omega = a(x, y)dx + b(x, y)dy$

Corollary 152 (Divergence Theorem) *Let $U \subset\in \mathbb{R}^3$ be an open bounded subset in $\mathbb{R}^3$ with ∂U a closed regular C^∞-smooth 2-surface. Let v be a C^1 vector field on (a neighbourhood of) $\overline{U}$. Then*

$$\int_{M=\overline{U}} \operatorname{div}(v) = \int_{S=\partial\overline{U}} i_v(\omega)$$

where $\omega = dx\wedge dy\wedge dz$ is the standard volume form on $\mathbb{R}^3$. In coordinates, if v has components $v^1 = v^1(x,y,z)$, $v^2 = v^2(x,y,z)$, $v^3 = v^3(x,y,z)$, the r.h.s

$$\int_{S=\partial\overline{U}} i_v(\omega) = \underbrace{\int_{S=\partial\overline{U}} v^1 dy\wedge dz + v^2 dz\wedge dx + v^3 dx\wedge dy}_{\int_S v\cdot d\vec{n} \text{ flux through } S}$$

Corollary 153 (Curl theorem) *Let $M^2 \subset \mathbb{R}^3$ be a compact regular oriented C^∞-smooth two-surface in $\mathbb{R}^3$ with boundary ∂M^2. Let v be a C^1 one-form in $\mathbb{R}^3$. Then*

$$\int_{M^2} \operatorname{curl}(v)\cdot d\vec{n} = \int_{M^2} \star\operatorname{rot}(v) = \int_{\partial M^2} \omega$$

where $\omega = \langle v, \cdot\rangle$, with $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^3$

Corollary 154 (Gradient Theorem) *Let $\sigma = (\gamma, [a,b])$, $\gamma : [a,b] \to \mathbb{R}^n$ be a $1-$cell in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function. Then*

$$\int_\sigma df = f(\sigma(b)) - f(\sigma(a))$$

Theorem 155 (Brouwer's fixed point theorem) *Let $\overline{B_r(0)} \subset \mathbb{R}^n$ be a closed ball in $\mathbb{R}^n$ around the origin and $f : \overline{B}_r(0) \to \overline{B}_r(0)$ be a C^2-smooth map. Then exists at least one fixed point $x_0 \in \overline{B_r(0)}$:*

$$f(x_0) = x_0$$